$SU(2)$ HIERARCHIES OF DILUTE LATTICE MODELS

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Abstract

The fusion procedure of dilute A_L models is constructed. It has been shown that the fusion rules have two types: $su(2)$ and $su(3)$. This paper is concerned with the $su(2)$ fusion rule mainly and the corresponding functional relations of commuting transfer matrices in the $su(2)$ fusion hierarchy are found. Specially, it has been found that the fusion hierarchy does not close. These two types of fusion generate different solvable models, but, they are not totally irrelevant. The $su(2)$ fusion of level 2 is equivalent to the $su(3)$ fusion of level $(1,1)$. According to this relationship the Bethe ansatz of fused model of level $(1,1)$ in $su(3)$ hierarchy has been represented by that of level 2 in $su(2)$ fusion hierarchy.

1 Introduction

In the family of restricted solid–on–solid (RSOS) models the dilute A_L lattice models [1] are very new two-dimensional solvable models and are the generalization of the ABF's RSOS models [2]. At criticality the dilute A_L lattice models [5] admit the D or E extension like the critical A–D–E models [6]. Thus the intertwiners have been constructed among these dilute A–D–E models [4].

The dilute A_L lattice models are built on the A_L Dynkin diagram with a loop at each spin node as shown in Figure 1. The face weights have the crossing symmetry and satisfy the inversion relations. The models at off criticality do not satisfy the $\mathbb{Z}_2$ symmetry[2] of

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[2]In fact, the $\mathbb{Z}_2$ symmetry is broken only for odd L.

Figure 1: The graph of adjacency condition of the dilute A_L models.

the adjacency condition graph Figure 1. Recent studies have shown that the dilute A_L models present many interesting aspects. In an appropriate regime the dilute A_3 model lies in the universality class of the Ising model in a magnetic field and gives the magnetic exponent $\delta = 15$ [7]. Also the A_3 model shows the E_8 scattering theory for massive excitations over the ground state [3, 8].

Fusion procedure is very useful in studying two-dimensional solvable vertex and face models [11, 12, 13, 15, 16, 17]. Essentially, fusion enables the construction of new solutions to the Yang-Baxter equations or star-triangle relation [9, 10] from a given fundamental solution. The fusion procedure of the dilute models has not been constructed. In this paper we construct the fusion of the dilute A models. This is completed by the method expressed in [17]. It can be shown that the models admit two types of fusion. The first one is $su(2)$-type and the other one is $su(3)$-type fusion. They are very different and employ different projectors for constructing fused face weights. We present the $su(2)$-fusion only in this paper. Differently the $su(3)$-fusion is more complicated and will be published elsewhere [14].

In next section we describe the dilute lattice models. The face weights satisfy a group of special properties which ensure that they can be taken as the elementary blocks for fusion. It is shown that there are two projectors, which generate two different types of fusion: $su(2)$ and $su(3)$ fusion based on the elementary models. In section 3 we give in detail the procedure for constructing the level 2 fused face weights of the $su(2)$-type. Then we construct the general procedure for finding the the $su(2)$ fused face weights of level n. This is accomplished by introducing "coordinates" on the independent paths of the fusion projectors. In section 4 we discuss the relationship between the $su(2)$ fusion and $su(3)$ fusion. It is shown that the fused model of level $(1,1)$ in the $su(3)$ fusion hierarchy is equivalent to the one of level 2 in the $su(2)$ fusion hierarchy. In section 5 we describe the $su(2)$ functional relations satisfied by the fused dilute A_L row transfer matrices. In section 6 the eigenvalues of transfer matrices of the $su(2)$ fusion hierarchy are discussed. In particular, the Bethe ansatz is also presented for the fused $(1,1)$ model of the $su(3)$ fusion hierarchy according to the relationship between these two fusions. In the final section a brief discussion is presented.

2 Elementary block

In this section we express the face weights of the dilute A_L models and list the properties of the face weighs. These properties are useful for constructing the fusion.

The states at adjacent sites of the dilute A_L square lattice must be adjacent on the graph in Figure 1. The face weights of the models not satisfying this adjacency condition for each pair of adjacent sites around a face vanish. The nonzero face weights of the dilute A_L models (off-critical) are given by [1]

$$
\begin{aligned}
W\left(\begin{matrix} a & a \\ a & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(6\lambda-u)\vartheta_1(3\lambda+u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} - \left(\frac{S(a+1)}{S(a)}\frac{\vartheta_4(2a\lambda-5\lambda)}{\vartheta_4(2a\lambda+\lambda)}\right. \\
&\quad \left. + \frac{S(a-1)}{S(a)}\frac{\vartheta_4(2a\lambda+5\lambda)}{\vartheta_4(2a\lambda-\lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a\pm1 & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
&\quad \left(\frac{\vartheta_4(\pm2a\lambda+3\lambda)\vartheta_4(\pm2a\lambda-\lambda)}{\vartheta_4^2(\pm2a\lambda+\lambda)}\right)^{1/2} \\
W\left(\begin{matrix} a\pm1 & a \\ a & a \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a & a \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(3\lambda-u)\vartheta_4(\pm2a\lambda+\lambda-u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\left(\begin{matrix} a & a \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) = \left(\frac{S(a\pm1)}{S(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_4(\pm2a\lambda-2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\left(\begin{matrix} a\mp1 & a \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \qquad (2.1) \\
W\left(\begin{matrix} a & a\mp1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= -\left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(3\lambda-u)\vartheta_1(\pm4a\lambda+2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)} \\
&\quad + \frac{S(a\pm1)}{S(a)}\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)}
\end{aligned}
$$

where the spectral parameter is u and the crossing parameter $\lambda = \frac{\pi}{4}\frac{L+2}{L+1}$ or $\frac{\pi}{4}\frac{L}{L+1}$. These $S(a)$ are given by

$$S(a) = (-1)^a\,\frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)}\,.$$

The elliptic functions $\vartheta_1(u), \vartheta_4(u)$ are the Jacobian ϑ-functions of *nome* p,

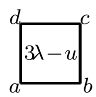
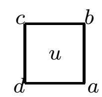
$$\vartheta_1(u) \quad = \quad 2p^{1/4}\sin u\prod_{n=1}^{\infty}(1-2p^{2n}\cos 2u+p^{4n})(1-p^{2n})$$

$$\vartheta_4(u) \;=\; \prod_{n=1}^{\infty}(1-2p^{2n-1}\cos 2u+p^{4n-2})(1-p^{2n})$$

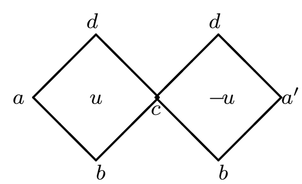
In critical limit $p \to 0$ the function $\vartheta_1(u)/\vartheta_1(\lambda) \sim \sin u/\sin\lambda$ and $\vartheta_4(u) \sim 1$. Under this limit these $S(a)$ are reduced to the elements S_a of the Perron-Frobenius eigenvectors S of the adjacency matrix A defined by the classical A_L Dynkin diagram with the elements

$$A_{a,b} = \begin{cases} 1, & |a-b| = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

The nonzero face weights can be graphically represented by the four spins surrounding square faces, or the graphs given by rotating clockwise these square faces through $\frac{\pi}{2}$. The face weights satisfy the following crossing symmetry

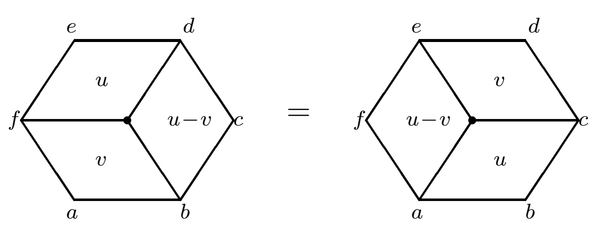
 $= \left(\dfrac{S(a)S(c)}{S(b)S(d)}\right)^{1/2}$  , (2.3)

the inversion relation of the form

$\sum_{c}$ 

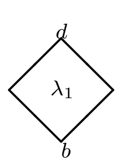
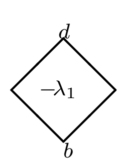
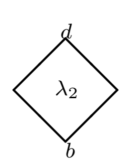
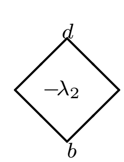
$$= \frac{\vartheta_1(2\lambda+u)\vartheta_1(3\lambda+u)\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1^2(2\lambda)\vartheta_1^2(3\lambda)} \tag{2.4}$$

and the star-triangle relation



(2.5)

where the sum is meant by the solid circle. The inversion relation vanishes for $u = \lambda_2 := 2\lambda$ or $u = \lambda_1 := 3\lambda$. This follows that the following two pairs of operators

 ,  and  , 

are singular. Each pair of them is orthogonal and can be taken as the projectors of

fusion. So two different types of fused models can be constructed from the elementary dilute models (2.1).

The projectors for the first fusion group are the face weights with the spectral parameters $\pm\lambda_1$ and they acquire the following properties

$$\begin{smallmatrix} & d & \\ a & \lambda_1 & c \\ & b\neq d & \end{smallmatrix} = 0\ , \tag{2.6}$$

$$\begin{smallmatrix} & b & \\ b\pm1 & \lambda_1 & c \\ & b & \end{smallmatrix} = A_{b\pm1,b}\left(\frac{S(b\pm1)}{S(b)}\right)^{1/2} \begin{smallmatrix} & b & \\ b & \lambda_1 & c \\ & b & \end{smallmatrix} \tag{2.7}$$

and inserting (2.7) into the inversion relation (2.4) we have

$$\begin{smallmatrix} & b & \\ a & -\lambda_1 & b \\ & b & \end{smallmatrix} = -\left(\frac{S(b-1)}{S(b)}\right)^{1/2} \begin{smallmatrix} & b & \\ a & -\lambda_1 & b-1 \\ & b & \end{smallmatrix} - \left(\frac{S(b+1)}{S(b)}\right)^{1/2} \begin{smallmatrix} & b & \\ a & -\lambda_1 & b+1 \\ & b & \end{smallmatrix} \quad . \tag{2.8}$$

For the second fusion group the projectors satisfy the following properties

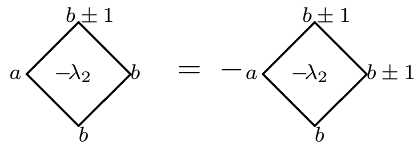
$$\begin{smallmatrix} & b & \\ a & \lambda_2 & a \\ & b\pm2 & \end{smallmatrix} = 0 \tag{2.9}$$

$$\begin{smallmatrix} & b & \\ b\pm1 & \lambda_2 & c \\ & b & \end{smallmatrix} = A_{b\pm1,b}\,\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\left(\frac{S(a\pm1,a)}{S(a,a)}\right)^{1/2} \begin{smallmatrix} & b & \\ b & \lambda_2 & c \\ & b & \end{smallmatrix} \tag{2.10}$$

$$\begin{smallmatrix} & b+1 & \\ b & \lambda_2 & c \\ & b & \end{smallmatrix} = \begin{smallmatrix} & b+1 & \\ b+1 & \lambda_2 & c \\ & b & \end{smallmatrix} \tag{2.11}$$

and inserting (2.10) and (2.11) into the inversion relation (2.4) we have

$$\left(\frac{S(b-1,b)}{S(b,b)}\right)^{1/2} \begin{smallmatrix} & b & \\ a & -\lambda_2 & b-1 \\ & b & \end{smallmatrix} + \left(\frac{S(b+1,b)}{S(b,b)}\right)^{1/2} \begin{smallmatrix} & b & \\ a & -\lambda_2 & b+1 \\ & b & \end{smallmatrix} = -\frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} \begin{smallmatrix} & b & \\ a & -\lambda_2 & b \\ & b & \end{smallmatrix} \tag{2.12}$$



(2.13)

where

$$
\begin{aligned}
S(a,b) &= S(a)f(a,b) && (2.14)\\
f(a,b) &= \left(\frac{\vartheta_4[2(a-b)b\lambda-\lambda]}{\vartheta_4[2(a-b)b\lambda]}\right)^{2|a-b|} \quad . && (2.15)
\end{aligned}
$$

The properties of the projectors for the critical dilute models can be given by taking critical limit $p \longrightarrow 0$ in (2.6)–(2.15).

The adjacency condition of the dilute A_L models can be represented by the classical A_L Dynkin diagram with a loop at each node. Each node a of the diagram has a coordination or valence val$(a) = 2, 3$, or

$$\mathrm{val}(a) = \sum_b (\delta_{a,b} + A_{a,b}) \ . \qquad (2.16)$$

The number of nonzero terms in (2.8) and (2.12) is given exactly by val(b). Specifically, we have the valence val$(b) = 3$ for most of the node b except for the endpoints 1 and L, which have val$(1) =$ val$(L) = 2$. For the ABF model [2] it has the classical A_L Dynkin diagram as the adjacency condition and the valence is less than 3: 1 for the endpoints $1, L$ and 2 for the other nodes. So the fusion for the dilute models is more complicated and proceeds differently.

The fusion for the second group has been given in [14]. In the following sections we will show that the first group fusion is of $su(2)$ type and the second one is of $su(3)$ type. We will describe the $su(2)$ fusion procedure in detail and also discuss the relation between these two fusions.

3 Fusion for shift λ_1

3.1 Admissibility

The adjacency matrix A of the models (2.1) satisfies the following fusion rule,

$$
\begin{aligned}
&A^{(n)}A^{(1)} = A^{(n-1)} + A^{(n+1)} \ , n = 1, 2, \cdots \\
&A^{(0)} = \mathbf{I}, \quad A^{(1)} = \mathbf{I} + A \qquad (3.1)
\end{aligned}
$$

where $\mathbf{I}$ is the unitary matrix and A is the adjacency matrix defined in (2.2). The fusion rule (3.1) in form likes the one of the classical A–D–E models and thus this fusion is of

$su(2)$ type. From (3.1) we can obtain

$$A^{(n)}A^{(n)} = A^{(n-1)}A^{(n+1)} + \mathbf{I} \quad . \tag{3.2}$$

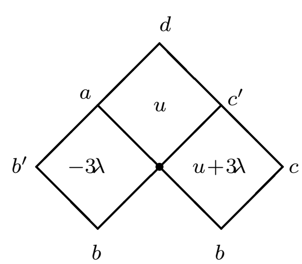
The matrices $A^{(n)}$ are the adjacency matrices of the fused face weights built using the projector with the spectral parameter shift λ_1. Unlike the classical A–D–E models the fusion of the dilute models with the shift λ_1 can go any higher fusion levels and hence do not have the closure condition. The elements of $A^{(n)}$ can in general be nonnegative integers greater than one. In this case we distinguish the edges of the adjacency diagram joining two given sites by bond variables $\alpha = 1, 2, \cdots$ If there is just one edge then the corresponding bond variable is $\alpha = 1$.

These fusion rules (3.1) and (3.2) can be extended to the level of transfer matrices, which give the functional relations for the transfer matrices and are constructed by the fusion procedure.

3.2 1 by 2 Fusion

We implement the elementary fusion of one by two block of face weights, which are the symmetric 1×2 fusion using the projector appeared in (2.8) and the antisymmetric 1×2 fusion using the projectors appeared in (2.7). Notice that in the level 2 fused models, the occurrence of bond variables on the edges of the symmetric fused face weights arises when both adjacent sites are the same spin with valence $\mathrm{val}(a) = 3$ or differ by 1 on the spin. The antisymmetric 1×2 fusion gives a trivial solution of the YBR because the nonzero fused face weights are all equal.

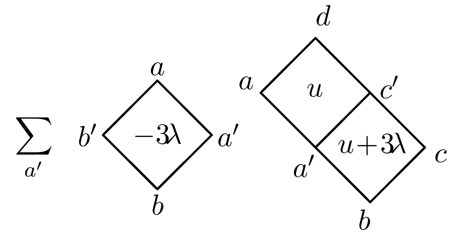
The symmetric fusion of level 2 can be expressed by the following objector.



(3.3)

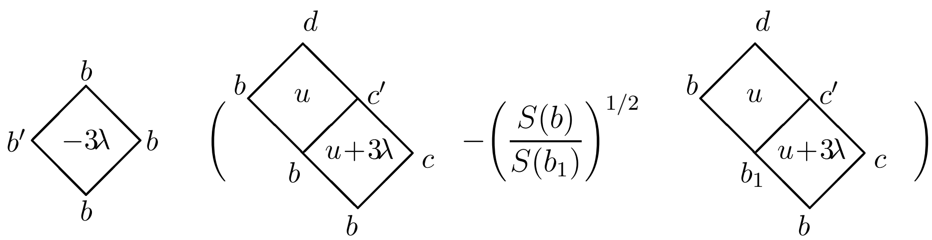
where the solid circle means sum over all possible spins, say, sum over a'. However, the summation vanishes if a' is not admissible to the neighbor spins a and b in the adjacency graph Figure 1. Thus using the property (2.8) the objector (3.3) can be expressed as the following cases.

1) $a \neq b$:



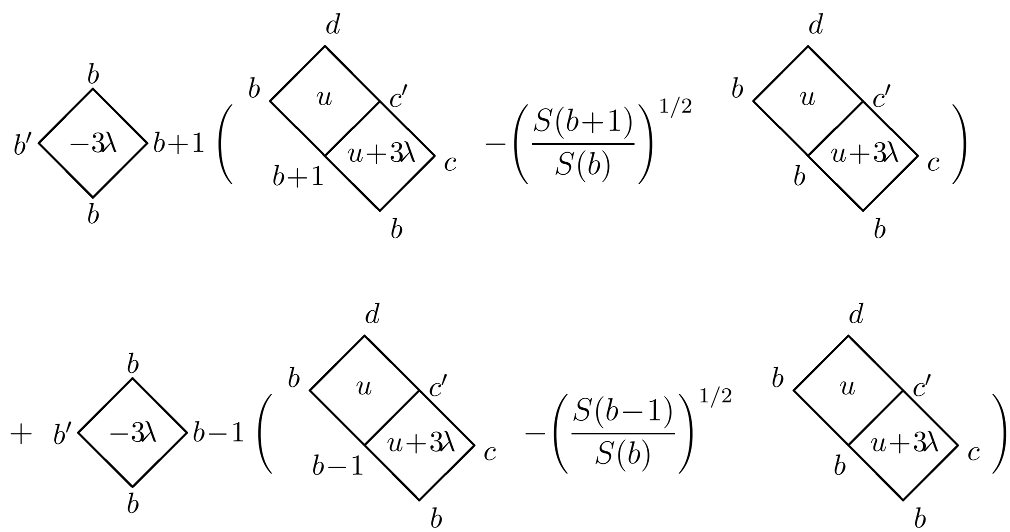
(3.4)

where $a' = a$ or b for $|a-b| = 1$ and $a' = (a+b)/2$ for $|a-b| = 2$.
2) $a = b$ and $\mathrm{val}(b) = 2$:



(3.5)

where $b_1 = 2$ for $b = 1$ and $b_1 = L-1$ for $b = L$.
3) $a = b$ and $\mathrm{val}(b) = 3$:



(3.6)

These expressions play exactly the role of the symmetric fusion of level 2. In (3.4) with $|a-b| = 1$ or (3.6) there are two independent terms classified by the projectors, which lead to two independent fused face weights. In (3.4) with $|a-b| = 2$ or (3.5) it has only one independent term according to the independent projector. Thus we have the following lemma.

Lemma 3.1 (Elementary Fusion) *If (a,b) and (d,c) are admissible edges at fusion level two we define the 1×2 fused weights for shift λ_1 by*

$$W_{1\times 2}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{a'} \phi^{\alpha}(a,a',b)\, W\left(\begin{matrix} d & c' \\ a & a' \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} c' & c \\ a' & b \end{matrix}\,\middle|\, u+\lambda_1\right) \tag{3.7}$$

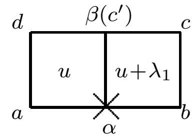


Figure 2: Elementary fusion of two faces. The cross denotes a symmetric sum labelled by $\alpha = 1, 2$ as defined in lemma 1. The other spins are fixed. If $c = d$ and val(c) = 3 we assume that $c' \neq c$. For clarity both the spin c' and the bond variable β are indicated.

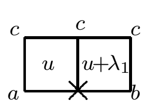
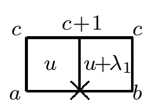
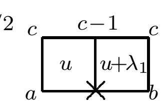
$$\phi^\alpha(a,a',b) = \begin{cases} (-)^{a+1-a'}\theta(a'-a)\left(\dfrac{S(a+1)}{S(a')}\right)^{1/2} & \textit{if } a=b,\ \alpha=1 \textit{ and val}(a)=3 \\ (-)^{a-1-a'}\theta(a-a')\left(\dfrac{S(a-1)}{S(a')}\right)^{1/2} & \textit{if } a=b,\ \alpha=2 \textit{ and val}(a)=3 \\ (-)^{a-a'}\left(\dfrac{S(a)}{S(a')}\right)^{1/2} & \textit{if } a=b \textit{ and val}(a)=2 \\ \delta_{a',a+\alpha-1} & \textit{if } a=b-1 \\ \delta_{a',b+\alpha-1} & \textit{if } a=b+1 \\ \delta_{a',(a+b)/2}\delta_{\alpha,1} & \textit{if } |a-b|=2 \end{cases} \tag{3.8}$$

where the step function $\theta(a<0)=0$ and $\theta(a\geq 0)=1$. The bond variable α (β) $=1$ for $|a-b|=2$ ($|c-d|=2$) or $a=b$ ($c=d$) with val(a) (val(c)) $=2$. The bond variable $\alpha=1$ and 2 for $a=b$ with val(a) $=3$ or $|a-b|=1$. $c'\neq c$ for $c=d$ and val(c) $=3$. Then it gives that (i)

$$\beta(c') = \begin{cases} 2 & \textit{if } c'=c-1=d-1 \textit{ and val}(c)=3 \\ 2 & \textit{if } c'=\textit{max}(c,d) \textit{ and } |c-d|=1 \\ 1 & \textit{otherwise.} \end{cases} \tag{3.9}$$

(ii) For all a,b,c,d the inversion relation (2.4) follows that $W_{1\times 2}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, 0\right) = 0.$

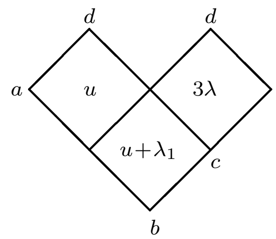
By (2.7) and the YBR (2.5) we have

$$\left(\frac{S(c)}{S(c+1)}\right)^{1/2}$$  $+$  $+\left(\dfrac{S(c-1)}{S(c+1)}\right)^{1/2}$  $= 0. \qquad (3.10)$

Thus it follows that the fused face weights satisfy the star-triangle relation (3.27) with $l = m = 1$ and $n = 2$.

The antisymmetric fusion is very simple to express and is given by the following

objector.



(3.11)

By the property (2.7) the objector (3.11) gives

Lemma 3.2 (Antisymmetric Fusion) *Put*

$$W_0\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \sum_{c'} \phi(d,c',c) W\left(\begin{matrix} d & c' \\ a & a' \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} c' & c \\ a' & b \end{matrix}\,\middle|\, u+\lambda_1\right) \tag{3.12}$$

$$\phi(d,c',c) = \begin{cases} 0 & \text{if } d \neq c \\ \left(\dfrac{S(c')}{S(c)}\right)^{1/2} & \text{if } d = c \end{cases} \tag{3.13}$$

It follows that

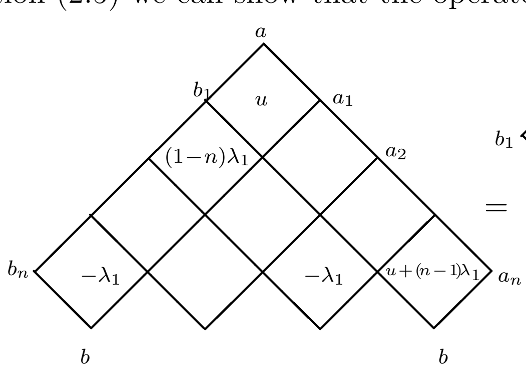
$$W_0\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = s_1^1(u) s_{-1}^1(u) s_{3/2}^1(u) s_{-3/2}^1(u) \delta_{a,b}\delta_{c,d} \;\;, \tag{3.14}$$

where

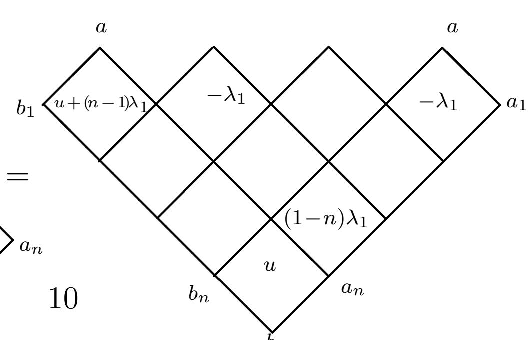
$$s_n^m(u) = \prod_{j=0}^{m-1} \left(\frac{\vartheta_1(u+(2n-3j)\lambda)}{\sqrt{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}} \right) , \tag{3.15}$$

3.3 Operator $P(n,u)$

The operator $P(n,-n\lambda_1)$ is the projector of level $n+1$ fusion, which element (a,b) is defined graphically by Fig.3 with $u=-n\lambda$. For more convenient let us consider $P(n,u)$ defined by Fig. 3. For $n=1$ it is the face weight of the elementary block and for $n=2$ it produces the 1 by 2 fusion presented in (3.4)–(3.6). With the help of star-triangle relation (2.5) we can show that the operator satisfies



$$=$$



(3.16)

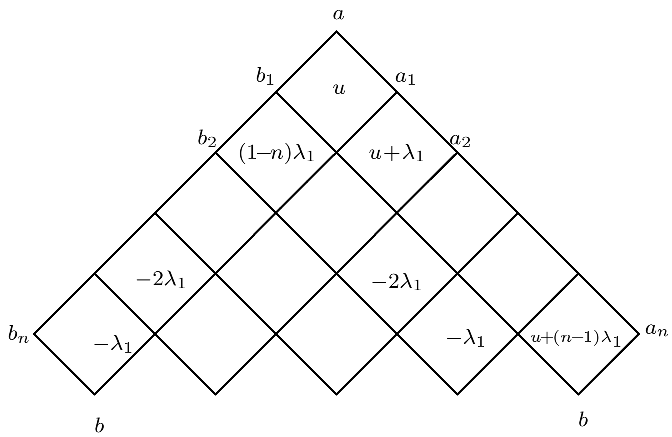


Figure 3: The difination of the $P(n,u)^{a,a_1,a_2,\cdots,b}_{a,b_1,b_2,\cdots,b}$.

Using (2.8) and the YBR (2.5) it is easy to see that any two adjacent faces with the spectral parameters $u+j\lambda_1$ and $u+(j-1)\lambda_1$ in Fig.3 can be considered as the symmetric 1 by 2 fusion. So the properties (3.10) imply

$$\begin{aligned} &A_{a_{j-1},a_{j-1}-1}\ P(n,u)^{(a,a_1,\cdots,a_{j-1},a_{j-1}-1,a_{j+1},\cdots,b)}_{(a,b_1,\cdots,b_{j-1},b_j,b_{j+1},\cdots,b)} \\ &\quad= -\frac{S(a_{j-1})}{S(a_{j-1}-1)}\ P(n,u)^{(a,a_1,\cdots,a_{j-1},a_{j-1},a_{j+1},\cdots,b)}_{(a,b_1,\cdots,b_{j-1},b_j,b_{j+1},\cdots,b)} \\ &\qquad -A_{a_{j-1},a_{j-1}+1}\frac{S(a_{j-1}+1)}{S(a_{j-1}-1)}P(n,u)^{(a,a_1,\cdots,a_{j-1},a_{j-1}+1,a_{j+1},\cdots,b)}_{(a,b_1,\cdots,b_{j-1},b_j,b_{j+1},\cdots,b)} \qquad \text{for } a_{j-1}=a_{j+1} \end{aligned} \tag{3.17}$$

Let the notation $p(a,b,n)$ represent the set of all allowed paths of n steps from a to b on the adjacent diagrams in Figure 1 and $P^{(n)}_{(a,b)}$ be the number of paths in the set $p(a,b,n)$. For convenience let $p(a,b,n)_i$ represent the i-th path in $p(a,b,n)$ and $p(a,b,n)_{i,j}$ be the j-th element of $p(a,b,n)_i$. So we can rewrite the elements of the projector $P(n-1,u)$ to be

$$P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_j}$$

The operator $P(n-1,u)$ is a suqare matrix and can be written in block diagonal form. The block $[P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_j}]$ is regular because of the properties (3.17), which give the eigenvectors of zero eigenvalues of the block. The number of non-zero eigenvalues of the block $[P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_j}]$, in fact, is given by $A^{(n)}_{(a,b)}$. Therefore the number of zero eigenvalues of the block then is given by $P^{(n)}_{(a,b)} - A^{(n)}_{(a,b)}$.

Two paths $p(a,b,n)_i$ and $p(a,b,n)_j$ may be related by the properties (3.17). If so we treat the paths $p(a,b,n)_i$ and $p(a,b,n)_j$ as dependent paths. Otherwise they are independent. Suppose there are $m^{(n)}_{(a,b)}$ independent realtions deriving from (3.17). So

there are $A^{(n)}_{(a,b)} = P^{(n)}_{(a,b)} - m^{(n)}_{(a,b)}$ independent paths in the set $p(a,b,n)$. We denote these independent paths by $\alpha(a,b,n)$, $\alpha = 1, 2, \cdots, A^{(n)}_{(a,b)}$ (there may be several ways to choose the independent paths, they give the equivalent fused models). The remaining paths are represented in terms of the independent paths

$$P(n-1,u)^{p(a,b,n)_i}_{\beta(a,b,n)} = \sum_{\alpha=1}^{A^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(n)}(a,b) P(n-1,u)^{\alpha(a,b,n)}_{\beta(a,b,n)}; \;\; i = 1, 2, \cdots, m^{(n)}_{(a,b)} \tag{3.18}$$

for any $\beta(a,b,n) \in \{\alpha(a,b,n) | \alpha = 1, 2, \cdots, A^{(n)}_{(a,b)}\}$. The value of $\phi^{(i,\alpha)}_{(n)}(a,b)$ takes zero if the path $p(a,b,n)_i$ is irrelevant to the path $\alpha(a,b,n)$ and takes nonzero value if dependent. According to (3.18) we can divide the set $p(a,b,n)$ into $A^{(n)}_{(a,b)}$ independent subsets defined by

$$p(n,a,\alpha,b) = \{(p(a,b,n)_i) | \phi^{(i,\alpha)}_{(n)}(a,b) \neq 0\}; \;\; \alpha = 1, 2, \cdots, A^{(n)}_{(a,b)}. \tag{3.19}$$

The first path in $p(n,a,\alpha,b)$ is the $\alpha(a,b,n)$, and i-th path is denoted by $p(n,a,\alpha,b)_i$ and $p(n,a,\alpha,b)_{i,j}$ is the j-th element of the path $p(n,a,\alpha,b)_i$. We call $\phi^{(i,\alpha)}_{(a,b,n)}$ as the coordinate of path $p(a,b,n)_i$ on the independent path $\alpha(a,b,n)$. By (3.16) it is obvious that

$$\phi^{(\alpha,\alpha)}_{(n)}(a,b) = \phi^{(i,i)}_{(n)}(a,b) = 1, \tag{3.20}$$

$$\phi^{(i,\alpha)}_{(n)}(a,b) = \phi^{(i,\alpha)}_{(n)}(b,a) \;, \tag{3.21}$$

$$\phi^{(\beta,\alpha)}_{(n)}(a,b) = \phi^{(\alpha,\beta)}_{(n)}(a,b) = 0 \quad \text{for } \alpha \neq \beta. \tag{3.22}$$

3.4 General Fusion

Let m and n be positive integers. Define

$$W_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) = \begin{array}{ccc} \mathrm{d} & \beta & \mathrm{c} \\ \mu & \boxed{\;\mathrm{u}\;} & \nu \\ \mathrm{a} & \alpha & \mathrm{b} \end{array} = \sum_{j=1}^{P^{(m)}_{(d,a)}} \phi^{(j,\mu)}_{(m)}(a,d) \sum_{\alpha_2,\cdots,\alpha_m}$$

$$\prod_{k=1}^{m} W_{1\times n}\left(\begin{array}{ccc|c} p(a,d,m)_{j,k+1} & \alpha_{k+1} & \nu(b,c,m)_{k+1} & \\ p(a,d,m)_{j,k} & \alpha_k & \nu(b,c,m)_k & u-(m-k)\lambda_1 \end{array}\right), \tag{3.23}$$

where $a = p(a,d,m)_{j,1}$, $b = \nu(b,c,m)_1$, $c = \nu(b,c,m)_{m+1}$, $d = p(a,d,m)_{j,m+1}$, $\alpha = \alpha_1$, $\beta = \alpha_{m+1}$, the summation α_k's is over $1, \cdots, A^{(n)}_{(p(a,d,m)_{j,k}, \nu(b,c,m)_k)}$. The $1\times n$ fusion in turn is given by

$$W_{1\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ a & \alpha & b & u \end{array}\right) =$$

$$\sum_{i=1}^{P^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(n)}(a,b) \prod_{k=1}^{n} W\left(\begin{array}{cc|c} \beta(d,c,n)_k & \beta(d,c,n)_{k+1} & \\ p(a,b,n)_{i,k} & p(a,b,n)_{i,k+1} & u+(k-1)\lambda_1 \end{array}\right) \;. \tag{3.24}$$

The fused face weights (3.23) defined here are similar to the ones of the critical D and E models [17]. In fact the following discussion proceeds as if critical D and E models. For the dilute A_L models we have the following lemma:

Lemma 3.3 *If the path $\beta(d,c,n)$ is replaced with its dependent path $p(n,d,\beta,c)_j$ then the fused weight*

$$W_{m\times n}\left(\begin{matrix} d & j & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{\beta'=1}^{A^{(n)}_{(d,c)}} \phi^{(j,\beta')}_{(d,c,n)}\, W_{m\times n}\left(\begin{matrix} d & \beta' & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \tag{3.25}$$

Similarly, if the path $\nu(b,c,m)$ is replaced by its dependent path $p(m,b,\nu,c)_j$ then

$$W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & j \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{\nu'=1}^{A^{(m)}_{(b,c)}} \phi^{(j,\nu')}_{(b,c,m)}\, W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu' \\ a & \alpha & b \end{matrix}\,\middle|\, u\right). \tag{3.26}$$

By applying YBE (2.5) to the tensor products of m by n elementary blocks and with the help of the Lemma 3.3 we can obtain the following theorem.

Theorem 3.4 *For a triple of positive integers m,n,l, the fused face weights (3.23) satisfy the following star-triangle relation*

$$\begin{aligned} &\sum_{(\eta_1,\eta_2,\eta_3)}\sum_g W_{m\times n}\left(\begin{matrix} e & \mu & d \\ \nu & & \eta_3 \\ f & \eta_1 & g \end{matrix}\,\middle|\, u\right) W_{m\times l}\left(\begin{matrix} d & \gamma & c \\ \eta_3 & & \beta \\ g & \eta_2 & b \end{matrix}\,\middle|\, u-v\right) W_{l\times n}\left(\begin{matrix} f & \eta_1 & g \\ \rho & & \eta_2 \\ a & \alpha & b \end{matrix}\,\middle|\, v\right) \\ &= \sum_{(\eta_1,\eta_2,\eta_3)}\sum_g W_{m\times n}\left(\begin{matrix} g & \eta_1 & c \\ \eta_3 & & \beta \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) W_{m\times l}\left(\begin{matrix} e & \eta_2 & g \\ \nu & & \eta_3 \\ f & \rho & a \end{matrix}\,\middle|\, u-v\right) W_{l\times n}\left(\begin{matrix} e & \mu & d \\ \eta_2 & & \gamma \\ g & \eta_1 & c \end{matrix}\,\middle|\, v\right) \end{aligned} \tag{3.27}$$

By construction it is obvious that $W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right)$ vanishes unless

$$\begin{aligned} A^{(n)}_{a,b} &\neq 0 \quad \text{and} \quad \alpha = 1,2,\cdots,A^{(n)}_{a,b} \\ A^{(n)}_{d,c} &\neq 0 \quad \text{and} \quad \beta = 1,2,\cdots,A^{(n)}_{d,c} \\ A^{(m)}_{d,a} &\neq 0 \quad \text{and} \quad \mu = 1,2,\cdots,A^{(m)}_{d,a} \\ A^{(m)}_{c,b} &\neq 0 \quad \text{and} \quad \nu = 1,2,\cdots,A^{(m)}_{c,b} \end{aligned}$$

where the fused adjacency matrices are given by (3.1). In particular the fused face weights exist for any higher fusion levels because we always have $A^{(n)} \neq 0$, $A^{(m)} \neq 0$ for $n,m>0$.

In conclusion we have constructed the fusion of the dilute A_L models for the shift λ_1. The Lemma 3.2 implies that the level 2 antisymmetric fusion is nothing but a trivial function of the spectral parameter and the crossing parameter. This behavior is similar to the fusion of the classical A-D-E models. The fusion of the dilute A_L models with the shift λ_1 is therefore thought as $su(2)$ type of fusion. This is consistent with the $su(2)$ adjacency fusion rule (3.1).

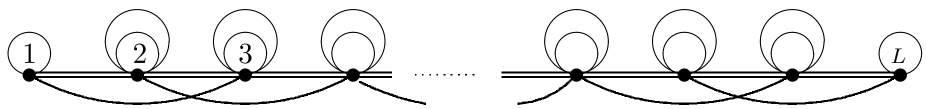


Figure 4: The graph of adjacency condition of the $su(2)$ fusion $A^{(2)}$ or the $su(3)$ fusion $A^{(1,1)}$.

4 Fusion (1,1) in su(3) hierarchy

The operators with the shift 2λ give the $su(3)$ type fusion of the dilute A_L models. The fusion procedure has been constructed in [14]. In this section we show some connections between the $su(2)$ fusion and the $su(3)$ fusion.

4.1 Adjacency conditions

The fusion with the shift λ_2 in structure is more interesting than the fusion with the shift λ_1. For the $su(3)$ fusion we need two numbers to label the fusion level. The adjacency matrices $A^{(n,m)}$ of the level (n,m) fused models are determined by the $su(3)$ fusion rules

$$A^{(n,m)}A^{(1,0)} = A^{(n+1,m)} + A^{(n-1,m+1)} + A^{(n,m-1)} \,, \tag{4.1}$$

$$A^{(0,0)} = \mathbf{I} \,, \quad A^{(1,0)} = \mathbf{I} + A \,, \quad A^{(n,m)} = A^{(m,n)} \tag{4.2}$$

$$A^{(n,m)} = 0 \ \text{ if } m,n < 0 \ \text{ or } \ n+m > 2L-1 \tag{4.3}$$

where $\mathbf{I}$ is the unitary matrix. A is the adjacency matrix (2.2). Like the $su(2)$ fusion, the elements of $A^{(n,m)}$ are nonnegative integers greater than or equal to one. The important point is that the $su(3)$ fusion rule has the closure condition $A^{(n,m)} = 0$ for $n+m = 2L$. Therefore the fusion hierarchies are truncated at the level $(m, 2L-m)$ with $m = 0, 1, \cdots, 2L$.

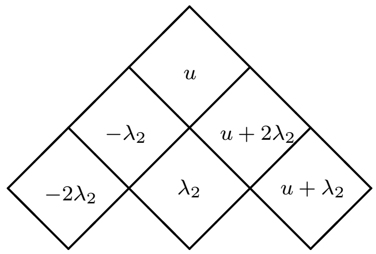
This $su(3)$ fusion rule is so different from previous $su(2)$ fusion rule. However they are not totally independent. Comparing them we find

$$A^{(2)} = A^{(1,1)} \,. \tag{4.4}$$

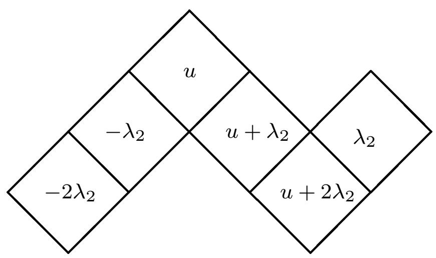
It follows that the $su(2)$ fusion of level 2 share the same adjacency condition with the $su(3)$ fusion of level $(1,1)$ (see Fig.4). In fact the fused face weights of these two fusions are the same up to some common functions.

4.2 1 by 3 Fusion: $(1, 1)$

The fusion of level $(1, 1)$ is given by the following object

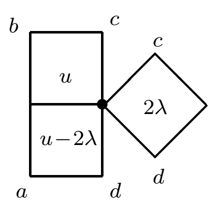


which can be written as

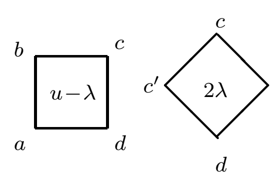


(4.1)

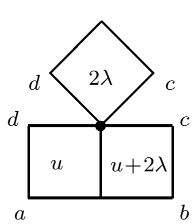
It can be seen that two projectors are used for the fusion (1,1). The bottom one (left side) is the fully symmetric projector and the top one (right side) is the fully antisymmetric projector, which is the elementary face with the spectral parameter $u = \lambda_2$. Instead of using the projectors we introduce two fusion parities $\psi^{\alpha}_{(1,1)}(d, d', c', c)$ on the top and $\phi^{\alpha}_{(1,1)}(a, a', b', b)$ on the bottom, which can be given by selecting the independent paths of these projectors. However, more simpler way is to do the antisymmetric fusion first on the top. It has been known that the antisymmetric fused weights are equivalent to the elementary face weights in following way [14],



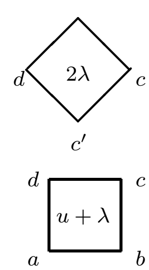
$\sim -s^1_0(u)s^1_{-5/2}(u)$



(4.2)



$\sim -s^1_1(u)s^1_{-3/2}(u)$



(4.3)

Where $c' = min(c, d)$ for $|c - d| = 1$ and $c' = c$ for $|c - d| = 0$. In (4.3) the projector graphically is the square face with anti-clockwise rotation through $\frac{\pi}{2}$. By these relations the objector (4.1) can be simply reduced to (3.3). So we have the following lemma.

Lemma 4.1 *Put*

$$W^{(1,1)}_{(1,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{a',b',c',d'} \psi^{\beta}_{(1,1)}(d,d',c',c)\,\phi^{\alpha}_{(1,1)}(a,a',b',b)$$

$$\times W\left(\begin{matrix} d & d' \\ a & a' \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} d' & c' \\ a' & b' \end{matrix}\,\middle|\, u{+}\lambda_2\right) W\left(\begin{matrix} c' & c \\ b' & b \end{matrix}\,\middle|\, u{+}2\lambda_2\right). \tag{4.4}$$

Then it follows that

(i) This fusion is equivalent to the su(2) fusion of level 2. Or disregarding the trivial gauge factors we have

$$W^{(1,1)}_{(1,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = -s^1_2(u)s^1_{-1/2}(u)W_{1\times 2}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \tag{4.5}$$

(ii) For all fixed a,b,c,d,α,β we have

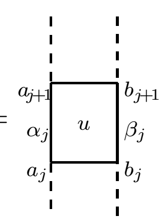
$$W^{(1,1)}_{(1,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = 0 \quad \textit{if } u = 0, \lambda, -2\lambda_2.$$

5 Su(2) Fusion hierarchy

The fusion rule (3.1) is the relations for the adjacency matrices of the fused models. We will see in this section that the theory carries over to the level of the row transfer matrix.

Suppose that $\mathbf{a}$ ($\boldsymbol{\alpha}$) and $\mathbf{b}$ ($\boldsymbol{\beta}$) are allowed spin (bond) configurations of two consecutive rows of an N (even) column lattice with periodic boundary conditions. The elements of the fused row transfer matrix $\mathbf{T}(u)$ are given by

$$\langle \boldsymbol{a},\boldsymbol{\alpha}|\mathbf{T}_{(m,n)}(u)|\boldsymbol{b},\boldsymbol{\beta}\rangle = \prod_{j=1}^{N}\sum_{\eta_j} W_{m\times n}\left(\begin{matrix} a_{j+1} & \eta_{j+1} & b_{j+1} \\ \alpha_j & & \beta_j \\ a_j & \eta_j & b_j \end{matrix}\,\middle|\, u\right) = \tag{5.1}$$



where $a_{N+1} = a_1$, $b_{N+1} = b_1$ and $\eta_{N+1} = \eta_1$. Specifically, the star-triangle relation (3.27) of the fused weights (3.23) imply the commutation relations

$$[\mathbf{T}_{(m,n)}(u), \mathbf{T}_{(m,\overline{n})}(v)] = 0. \tag{5.2}$$

Thus if fix m we have the hierarchy of commuting families of transfer matrices. The fusion procedure implies various relations among these transfer matrices. We summarize them in following theorems.

Theorem 5.1 ($su(2)$ Fusion Hierarchy) *Let us define*

$$\begin{aligned} \boldsymbol{T}_k^{(n)} &= \boldsymbol{T}_{(m,n)}(u+3k\lambda) \\ \boldsymbol{T}^{(n)} &= 0 \quad \textit{if } n<0 \textit{ or } m<0 \\ \boldsymbol{T}^{(0)} &= \boldsymbol{I} \\ f_n^m &= \left[s_{3n/2+1}^m(u)s_{3n/2-1}^m(u)s_{3(n+1)/2}^m(u)s_{3(n-1)/2}^m(u)\right]^N \end{aligned} \tag{5.3}$$

Then for $n \geq 0$

$$\boldsymbol{T}_0^{(n)}\ \boldsymbol{T}_n^{(1)} = \ \boldsymbol{T}_0^{(n+1)} + f_{n-1}^m\ \boldsymbol{T}_0^{(n-1)} \tag{5.4}$$

without closure condition.

Starting with the fusion hierarchy we can easily derive new functional equations

$$\boldsymbol{T}_0^{(n)}\ \boldsymbol{T}_1^{(n)} = \ \boldsymbol{I}\prod_{k=0}^{n-1} f_k^m + \ \boldsymbol{T}_0^{(n+1)}\ \boldsymbol{T}_1^{(n-1)} \tag{5.5}$$

Then it is easy to see the following theorem.

Theorem 5.2 ($su(2)$ TBA) *If we define*

$$\boldsymbol{t}_0^0 \ = \ 0 \tag{5.6}$$

$$\boldsymbol{t}_0^n \ = \ \frac{\boldsymbol{T}_0^{(n+1)}\ \boldsymbol{T}_1^{(n-1)}}{\boldsymbol{I}\prod_{k=0}^{n-1} f_k^m}\ , \tag{5.7}$$

then it follows that $su(2)$ TBA-like equations

$$\boldsymbol{t}_0^n\ \boldsymbol{t}_1^n = (\ \boldsymbol{I} + \ \boldsymbol{t}_0^{n+1})(\ \boldsymbol{I} + \ \boldsymbol{t}_1^{n-1}) \tag{5.8}$$

without closure condition.

The fusion hierarchy (5.4) and the fusion rule (3.1) are similar in form. They can be associated with affine $su(2)$ and there is an one-to-one correspondence between $\boldsymbol{T}^{(n)}$ or $A^{(n)}$ and a Young diagram. Let us represent $\boldsymbol{T}^{(n)}$ or $A^{(n)}$ by a Young diagram with n blocks in one row. Identify any Young diagram with two rows as an one row Young diagram by subtracting the columns of length 2. Then the fusion hierarchy (5.4) or the fusion rule (3.1) can be represented graphically by

$$\begin{aligned} &\underbrace{\boxed{}}_{n} \otimes \boxed{} \\ &= \underbrace{\boxed{}}_{n-1} \oplus \underbrace{\boxed{}}_{n+1} \end{aligned} \tag{5.9}$$

and the functional relations (3.2) or (5.5) are represented by

$$
\begin{aligned}
& \underbrace{\boxed{}}_{n} \otimes \underbrace{\boxed{}}_{n} \\
= & \underbrace{\boxed{}}_{n-1} \otimes \underbrace{\boxed{}}_{n+1} \oplus \bullet
\end{aligned}
\tag{5.10}
$$

where the dot means the identity matrix.

6 Bethe Ansatz

The eigenvalues $\Lambda_{(1,0)}(u)$ and the Bethe ansatz equations of the row transfer matrices $\boldsymbol{T}(u)$ has been given in [3]. Obviously, the eigenvalues and the Bethe ansatz equations can be extended to the transfer matrix $\boldsymbol{T}_{(m,1)}(u)$ and they are given by

$$
\begin{aligned}
\Lambda_{(m,1)}(u) \;\;=\;\; & \omega\,[s^m_{-1}(u)s^m_{-3/2}(u)]^N \frac{Q^{(m)}(u+\lambda)}{Q^{(m)}(u-\lambda)} + \omega^{-1}\,[s^m_0(u)s^m_{-1/2}(u)]^N \frac{Q^{(m)}(u-4\lambda)}{Q^{(m)}(u-2\lambda)} \\
& + [(-1)^m s^m_0(u)s^m_{-3/2}(u)]^N \frac{Q^{(m)}(u)Q^{(m)}(u-3\lambda)}{Q^{(m)}(u-\lambda)Q^{(m)}(u-2\lambda)}
\end{aligned}
\tag{6.11}
$$

where

$$
Q^{(m)}(u) = \prod_{j=1}^{mN} \vartheta_1(u - iu_j) \tag{6.12}
$$

and the zeros $\{u_j\}$ satisfy the Bethe ansatz equations

$$
\omega\left[\frac{s^m_{1/2}(iu_j)}{s^m_{-1/2}(iu_j)}\right]^N \;=\; (-1)^{mN+1} \prod_{k=1}^{mN} \frac{\vartheta_1(iu_j - iu_k + 2\lambda)\vartheta_1(iu_j - iu_k - \lambda)}{\vartheta_1(iu_j - iu_k - 2\lambda)\vartheta_1(iu_j - iu_k + \lambda)} \tag{6.13}
$$

with $j = 1, \cdots, N$ and $\omega = \exp(i\pi\ell/(L+1))$, $\ell = 1, \cdots, L$. The Bethe ansatz equations ensure that the eigenvalues $\Lambda_{(m,1)}(u)$ are entire functions of u. Inserting the solution $\Lambda_{(m,1)}(u)$ into the functional relations (5.4) we can obtain the eigenvalues of the transfer matrices of whole hierarchy. In particular, the eigenvalues of $\boldsymbol{T}_{(2,2)}$ are given by

$$
\begin{aligned}
\Lambda_{(2,2)}(u) \;\;=\;\; & \Lambda_{(2,1)}(u)\Bigg(\omega\,[s^2_0(u)s^2_{1/2}(u)]^N \frac{Q^{(2)}(u+4\lambda)}{Q^{(2)}(u+2\lambda)} \\
& \qquad + [s^2_{3/2}(u)s^2_0(u)]^N \frac{Q^{(2)}(u+3\lambda)Q^{(2)}(u)}{Q^{(2)}(u+2\lambda)Q^{(2)}(u+\lambda)}\Bigg) \\
& + \omega^{-1}\,[s^2_1(u)s^2_{3/2}(u)]^N \frac{Q^{(2)}(u-\lambda)}{Q^{(2)}(u+\lambda)}
\end{aligned}
$$

$$
\times \left(\omega^{-1} \left[s_0^2(u) s_{-1/2}^2(u) \right]^N \frac{Q^{(2)}(u-4\lambda)}{Q^{(2)}(u-2\lambda)} \right.
$$

$$
\left. + \left[s_0^2(u) s_{-3/2}^2(u) \right]^N \frac{Q^{(2)}(u) Q^{(2)}(u-3\lambda)}{Q^{(2)}(u-\lambda) Q^{(2)}(u-2\lambda)} \right) \tag{6.14}
$$

This represents also the eigenvalues of the transfer matrix

$$
\langle \boldsymbol{a}, \boldsymbol{\alpha} | \mathbf{T}^{(1,1)}_{(1,1)}(u) | \boldsymbol{b}, \boldsymbol{\beta} \rangle = \prod_{j=1}^{N} \sum_{\eta_j} W^{(1,1)}_{(1,1)} \left(\begin{array}{ccc|c} a_{j+1} & \eta_{j+1} & b_{j+1} & \\ \alpha_j & & \beta_j & u \\ a_j & \eta_j & b_j & \end{array} \right) \tag{6.15}
$$

for the fusion level $(1,1) \times (1,1)$ in $su(3)$ fusion hierarchy.

7 Discussion

We have presented the $su(2)$ fusion procedure for the dilute A_L models. The fusion of critical dilute D and E models can be implemented in a similar way [19]. The functional relations of the $su(2)$ fusion hierarchy do not close and the fusion exists for any higher level. This likes the fusion hierarchy of the vertex models. However the dilute models are the restricted SOS models. We have infinite number of the fused models in this $su(2)$ fusion hierarchy and all these fused models are the restricted SOS models. This behavior is generally not common for the restricted SOS models. So we need to study further to understand this fusion hierarchy. Particularly, it is interesting to find the finite-size corrections of the transfer matrices of the $su(2)$ fused models. In the $su(3)$ hierarchy the finite-size corrections of the dilute models have been studied [18].

As a possible way we may treat the $su(2)$ fusion hierarchy like the spin 1 representation of $su(2)$. This however is allowed for the adjacency matrices. Attach the Young diagrams to the adjacency matrix $A^{(1)}$ and $A^{(0)} = \boldsymbol{I}$ in following ways,

$$
\tilde{A}^{(2)} := A^{(1)} \quad \sim \quad \Box\!\Box \;=\; \boxed{} \tag{7.16}
$$

$$
\tilde{A}^{(0)} := A^{(0)} \quad \sim \quad \begin{array}{c} \Box \\[-4pt] \Box \end{array} \;=\; \bullet \tag{7.17}
$$

We can have the following $su(2)$ fusion rule

$$
\tilde{A}^{(2n)} \tilde{A}^{(2)} = \tilde{A}^{(2n-2)} + \tilde{A}^{(2n)} + \tilde{A}^{(2n+2)} \tag{7.18}
$$

with

$$
\tilde{A}^{(2L)} = Y \ , \tag{7.19}
$$

where $n = 1, 2, \cdots, L-1$ and Y is the spin reversal operator or $Y_{a,b} = \delta_{a,L-b}$. This fusion rule is of $su(2)$ type, but, differs to the $su(2)$ fusion rule (3.1). We may derive the new fusion rule (7.18) with Young diagrammatic methods as follows.

$$\begin{aligned} &\underbrace{\boxed{}}_{2n} \otimes \boxed{} \\ &= \underbrace{\boxed{}}_{2n-2} \oplus \underbrace{\boxed{}}_{2n} \oplus \\ &\quad \underbrace{\boxed{}}_{2n+2} \end{aligned} \tag{7.20}$$

where we have used the graphical representation

$$\tilde{A}^{(2n)} \quad \sim \quad \underbrace{\boxed{}}_{2n} \ . \tag{7.21}$$

This new fusion rule shares the $\mathbb{Z}_2$ symmetry

$$\tilde{A}^{(2L-2n)} = Y\tilde{A}^{(2n)} = \tilde{A}^{(2n)}Y \ , \quad n = 0, 1, \cdots, L \ . \tag{7.22}$$

Therefore it is easy to see that at each level $2kL+1$ for any positive integer k it starts to repeat the fusion rule (7.18) again and thus this fusion rule is truncated. Here it has been shown that this procedure works for the adjacency matrices. It is not clear how it works for the transfer matrices.

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